

# On Reichenbach's causal betweenness

Vášek Chvátal

*Canada Research Chair in Combinatorial Optimization  
Department of Computer Science and Software Engineering  
Concordia University  
Montreal, Quebec H3G 1M8, Canada*

Baoyindureng Wu

*College of Mathematics and System Sciences  
Xinjiang University  
Urumqi 830046, PR China*

## Abstract

We characterize, by easily verifiable properties, abstract ternary relations isomorphic to the causal betweenness introduced by Hans Reichenbach.

## 1 Introduction

A *finite probability space* is an ordered pair  $(S, p)$  where  $S$  is a finite set and  $p : S \rightarrow [0, 1]$  is a function such that

$$\sum_{s \in S} p(s) = 1.$$

The set  $S$  is called a *sample space* and its subsets are called *events*; when  $A$  and  $B$  are events,  $AB$  denotes  $A \cap B$ . The *probability*  $P(A)$  of event  $A$  is defined by

$$P(A) = \sum_{s \in A} p(s);$$

when  $A$  and  $B$  are events with  $P(B) > 0$ , the *conditional probability*  $P(A|B)$  of  $A$  given  $B$  is defined by

$$P(A|B) = \frac{P(AB)}{P(B)}.$$

Hans Reichenbach (1956, p. 190) defined an event  $B$  to be *causally between* events  $A$  and  $C$  if the following relations hold:

$$P(AC) > P(A) \cdot P(C), \quad (1)$$

$$P(C|B) > P(C|A), \quad (2)$$

$$P(A|B) > P(A|C), \quad (3)$$

$$P(AC|B) = P(A|B) \cdot P(C|B), \quad (4)$$

and

$$P(B - A) > 0, P(B - C) > 0. \quad (5)$$

(The conditional probabilities in (2) – (4) are well-defined: (1) guarantees that  $P(A) > 0$ ,  $P(C) > 0$  and (5) guarantees that  $P(B) > 0$ . In terms of Reichenbach (1956, p. 189, pp. 201 – 205) equation (4) means that  $B$  *screens off*  $A$  from  $C$ .) Following Reichenbach's work, causal betweenness was considered by von Bretzel (1977), Ellett and Ericson (1986), Dowe (1992), Weber (1997), Korb (1999), and others.

Given a set  $X$  of events in a finite probability space, we let  $\mathcal{CB}(X)$  denote the set of all ordered triples  $(A, B, C)$  such that  $A, B, C$  are events in  $X$  with properties (1) – (5). We say that a ternary relation  $\mathcal{B}$  on a finite ground set is an *abstract causal betweenness* if, and only if, there is a set  $X$  of events in a finite probability space such that  $\mathcal{CB}(X)$  is isomorphic to  $\mathcal{B}$ . Our Theorem 1 characterizes abstract causal betweennesses by easily verifiable properties.

We call a ternary relation  $\mathcal{B}$  a *betweenness* if

$$(A, B, C) \in \mathcal{B} \Rightarrow A, B, C \text{ are all distinct,}$$

$$(A, B, C) \in \mathcal{B} \Rightarrow (C, B, A) \in \mathcal{B},$$

$$(A, B, C) \in \mathcal{B} \Rightarrow (C, A, B) \notin \mathcal{B}.$$

The familiar concept of betweenness in Euclidean geometry generalizes in diverse branches of mathematics to betweennesses  $\mathcal{B}$  with the property

$$(ABC), (ADB) \in \mathcal{B} \Rightarrow (ADC) \in \mathcal{B}. \quad (6)$$

These relations include *metric betweenness*, *lattice betweenness* in modular lattices, and *algebraic betweenness* (see, for instance, Pitcher and Smiley (1942),

Smiley (1943), Hashimoto (1958), Bumcrot (1964)). Our Corollary 1 asserts that every betweenness  $\mathcal{B}$  with property (6) is an abstract causal betweenness.

In Reichenbach’s investigations, events occur in time and time order is reduced to causal order (Reichenbach 1956, p. 24). An event that is causally between events  $A$  and  $C$  does not necessarily occur between  $A$  and  $C$  (after  $A$  and before  $C$ , or else after  $C$  and before  $A$ ). To elaborate on this point, we say that a ternary relation  $\mathcal{B}$  on a finite set is *totally orderable* if, and only if, there is a mapping  $t$  from the ground set of  $\mathcal{B}$  to a set with a total order  $\prec$  such that

$$(A, B, C) \in \mathcal{B} \Rightarrow (t(A) \prec t(B) \prec t(C) \text{ or } t(C) \prec t(B) \prec t(A)).$$

Reichenbach (1956, p. 192) pointed out that  $\mathcal{B}$  is not totally orderable when it includes the ordered triples  $(A_1, A_2, A_3)$ ,  $(A_1, A_2, A_4)$ ,  $(A_4, A_2, A_3)$ ; these three triples, along with their reversals  $(A_3, A_2, A_1)$ ,  $(A_4, A_2, A_1)$ ,  $(A_3, A_2, A_4)$ , constitute an abstract causal betweenness. Therefore not every abstract causal betweenness is totally orderable.

Opatrný (1979) proved that recognizing totally orderable ternary relations is hard: the problem is  $\mathcal{NP}$ -complete. (Readers unfamiliar with the notion of  $\mathcal{NP}$ -completeness are referred to the monograph of Garey & Johnson (1979).) The problem does not get any easier when its input is restricted to abstract causal betweennesses: our Corollary 2 asserts that every totally orderable betweenness is an abstract causal betweenness, and so testing an arbitrary ternary relation  $\mathcal{B}$  for total orderability reduces to testing an abstract causal betweenness for total orderability. (We first test  $\mathcal{B}$  for being an abstract causal betweenness; Theorem 1 shows how to carry out this test easily; if  $\mathcal{B}$  fails it, then Corollary 2 guarantees that  $\mathcal{B}$  is not totally orderable.)

## 2 Results

With each betweenness  $\mathcal{B}$  on a ground set  $X$ , we associate a directed graph  $G(\mathcal{B})$ . Its vertices are all two-point subsets of  $X$ ; its edges are all ordered pairs  $(\{A, B\}, \{A, C\})$  such that  $(ABC) \in \mathcal{B}$ . These graphs may contain directed cycles: if  $\{(DAB), (DBC), (DCA)\} \subseteq \mathcal{B}$ , then  $G(\mathcal{B})$  contains the

directed cycle

$$\{D, A\} \rightarrow \{D, B\} \rightarrow \{D, C\} \rightarrow \{D, A\},$$

if  $\{(CAB), (DBC), (ACD), (BDA)\} \subseteq \mathcal{B}$ , then  $G(\mathcal{B})$  contains the directed cycle

$$\{A, B\} \rightarrow \{B, C\} \rightarrow \{C, D\} \rightarrow \{D, A\} \rightarrow \{A, B\},$$

and so on.

**Theorem 1** *A ternary relation  $\mathcal{B}$  on a finite set is an abstract causal betweenness if and only if  $\mathcal{B}$  is a betweenness and  $G(\mathcal{B})$  contains no directed cycle.*

**Corollary 1** *Every betweenness  $\mathcal{B}$  with the property*

$$(ABC), (ADB) \in \mathcal{B} \Rightarrow (ADC) \in \mathcal{B}.$$

*is an abstract causal betweenness.*

**Corollary 2** *Every totally orderable betweenness is an abstract causal betweenness.*

### 3 Proofs

**Proof of Theorem 1.**

The “if” part: Consider an arbitrary betweenness  $\mathcal{B}$  on a ground set  $X$  such that the directed graph  $G(\mathcal{B})$  contains no directed cycle. Without loss of generality,  $X = \{1, 2, \dots, m\}$  for some positive integer  $m$ . We shall construct a finite probability space and events  $E_1, E_2, \dots, E_m$  in this space in such a way that  $E_j$  is causally between  $E_i$  and  $E_k$  if and only if  $(i, j, k) \in \mathcal{B}$ .

The construction proceeds in two stages. First, we choose an arbitrarily small positive  $\varepsilon$  and we construct functions

$$\begin{aligned} \beta : \{W : W \subseteq X, |W| = 2\} &\rightarrow (0.25, 0.25 + \varepsilon), \\ \gamma : \{W : W \subseteq X, |W| = 3\} &\rightarrow (0.125, 0.125 + \varepsilon) \end{aligned}$$

such that  $(i, j, k) \in \mathcal{B}$  if and only if

$$\begin{aligned}\gamma(\{i, j, k\}) &= 2\beta(\{i, j\})\beta(\{j, k\}), \\ \beta(\{i, j\}) &> \beta(\{i, k\}), \\ \beta(\{j, k\}) &> \beta(\{i, k\}).\end{aligned}$$

Then we construct a finite probability space and events  $E_1, E_2, \dots, E_m$  in this space in such a way that

$$\begin{aligned}\mathbb{P}(E_i) &= 0.5 && \text{for all subscripts } i, \\ \mathbb{P}(E_i E_j) &= \beta(\{i, j\}) && \text{for all choices of distinct subscripts } i, j, \\ \mathbb{P}(E_i E_j E_k) &= \gamma(\{i, j, k\}) && \text{for all choices of distinct subscripts } i, j, k.\end{aligned}$$

In the first stage, we choose  $\varepsilon$  and  $\delta$  so that

$$0 < \varepsilon < m^{-2}4^{-m}, \quad 0 < \delta < m^{-2}\varepsilon.$$

Since  $G(\mathcal{B})$  contains no directed cycle, there is a mapping

$$\rho : \{W \subset X : |W| = 2\} \rightarrow \{1, 2, \dots, m(m-1)/2\}$$

such that

$$(i, j, k) \in \mathcal{B} \Rightarrow \rho(\{i, j\}) > \rho(\{i, k\}).$$

We set

$$\beta(\{i, j\}) = 0.25 + \delta\rho(\{i, j\})$$

for all choices of distinct subscripts  $i, j$ . For all  $(i, j, k)$  in  $\mathcal{B}$ , we set

$$\gamma(\{i, j, k\}) = 2\beta(\{i, j\})\beta(\{j, k\});$$

if  $i, j, k$  are distinct subscripts such that none of  $(i, j, k)$ ,  $(j, k, i)$ ,  $(k, i, j)$  is in  $\mathcal{B}$ , then we choose  $\gamma(\{i, j, k\})$  in the interval  $(0.125, 0.125 + \varepsilon)$  and distinct from all three of

$$2\beta(\{i, j\})\beta(\{j, k\}), \quad 2\beta(\{j, k\})\beta(\{k, i\}), \quad 2\beta(\{k, i\})\beta(\{i, j\}).$$

The upper bound on  $\delta$  guarantees that

$$0.25 < \beta(\{i, j\}) < 0.25 + \varepsilon/2$$

for all choices of distinct subscripts  $i, j$  and that

$$0.125 < \gamma(\{i, j, k\}) < 0.125 + \varepsilon$$

for all choices of distinct subscripts  $i, j, k$ .

In the second stage, we begin with sample space  $\{0, 1\}^m$  and we set

$$E_i = \{s \in \{0, 1\}^m : s_i = 1\} \quad (i = 1, 2, \dots, m).$$

For each subset  $W$  of  $X$ , let  $\chi^W$  denote the vector in  $\{0, 1\}^m$ , whose  $i$ -th coordinate is 1 if and only if  $i \in W$ . We will complete the proof by exhibiting a function  $p : \{0, 1\}^m \rightarrow (0, 1)$  such that

$$\sum(p(\chi^W) : i, j, k \in W) = \gamma(\{i, j, k\}) \quad (7)$$

for all choices of distinct subscripts  $i, j, k$ ,

$$\sum(p(\chi^W) : i, j \in W) = \beta(\{i, j\}) \quad (8)$$

for all choices of distinct subscripts  $i, j$ ,

$$\sum(p(\chi^W) : i \in W) = 0.5 \quad (9)$$

for all subscripts  $i$ , and

$$\sum_W p(\chi^W) = 1. \quad (10)$$

For this purpose, we set first

$$p(\chi^W) = 2^{-m} \quad \text{whenever } |W| \geq 4$$

and

$$p(\chi^{\{i, j, k\}}) = 2^{-m} + (\gamma(\{i, j, k\}) - 0.125)$$

to satisfy (7) for all choices of distinct subscripts  $i, j, k$ , then

$$\begin{aligned} p(\chi^{\{i, j\}}) &= 2^{-m} + (\beta(\{i, j\}) - 0.25) \\ &\quad - \sum(p(\chi^W) - 2^{-m} : W \supset \{i, j\}, |W| = 3) \end{aligned}$$

to satisfy (8) for all choices of distinct subscripts  $i, j$ , then

$$p(\chi^{\{i\}}) = 2^{-m} - \sum(p(\chi^W) - 2^{-m} : W \ni i, 2 \leq |W| \leq 3)$$

to satisfy (9) for all subscripts  $i$ , and finally

$$p(\chi^\emptyset) = 1 - \sum_{W \neq \emptyset} p(\chi^W).$$

to satisfy (10). Now

$$2^{-m} < p(\chi^{\{i,j,k\}}) < 2^{-m} + \varepsilon$$

for all choices of distinct subscripts  $i, j, k$ ,

$$2^{-m} - m\varepsilon < p(\chi^{\{i,j\}}) < 2^{-m} + \varepsilon$$

for all choices of distinct subscripts  $i, j$ ,

$$2^{-m} - m^2\varepsilon < p(\chi^{\{i\}}) < 2^{-m} + m^2\varepsilon$$

for all subscripts  $i$ , and so the upper bound on  $\varepsilon$  guarantees that all  $p(\chi^W)$  are positive.

The “only if” part: Consider an arbitrary set  $X$  of events in a finite probability space. Reichenbach (1956, p. 191) proved that  $\mathcal{CB}(X)$  is a betweenness. We shall reproduce his argument here and we shall show that  $G(\mathcal{CB}(X))$  contains no directed cycle.

To prove that  $\mathcal{CB}(X)$  is a betweenness, consider a triple  $(A, B, C)$  of events that satisfy (1) – (5). Assumption (5) guarantees  $B \neq A$  and  $B \neq C$ ; assumption (2) guarantees  $P(C|A) < 1$ , which implies  $C \neq A$ ; now  $A, B, C$  are all distinct. Since the set of assumptions (1) – (5) is invariant under the switch  $A \leftrightarrow C$ , the triple  $(C, B, A)$  satisfies them in place of  $(A, B, C)$ . Finally, the triple  $(C, A, B)$  fails to satisfy (4) in place of  $(A, B, C)$  since (4) and (2) imply

$$P(CB|A) = P(C|B) \cdot P(B|A) > P(C|A) \cdot P(B|A).$$

To see that  $G(\mathcal{CB}(X))$  contains no directed cycle, observe that (3) implies

$$(A, B, C) \in \mathcal{B} \Rightarrow \frac{P(AB)}{P(A) \cdot P(B)} > \frac{P(AC)}{P(A) \cdot P(C)}.$$

**Proof of Corollary 1.** Writing  $\sigma(A, B)$  for the number of elements  $D$  such that  $(ADB) \in \mathcal{B}$ , observe that (6) implies

$$(A, B, C) \in \mathcal{B} \Rightarrow \sigma(A, B) < \sigma(A, C),$$

and so  $G(\mathcal{B})$  contains no directed cycle.

**Proof of Corollary 2.** Consider an arbitrary betweenness  $\mathcal{B}$  on a finite set along with a mapping  $t$  from the ground set of  $\mathcal{B}$  to the set of real numbers such that

$$(A, B, C) \in \mathcal{B} \Rightarrow (t(A) < t(B) < t(C) \text{ or } t(C) < t(B) < t(A)).$$

Writing  $\tau(A, B) = |t(A) - t(B)|$ , observe that

$$(A, B, C) \in \mathcal{B} \Rightarrow \tau(A, B) < \tau(A, C),$$

and so  $G(\mathcal{B})$  contains no directed cycle.

### Acknowledgment

This note was written while the second author was visiting the first author in ConCoCO (Concordia Computational Combinatorial Optimization Laboratory). The research was undertaken, in part, thanks to funding from the Canada Research Chairs Program and by the China Scholarship Council.

## References

- von Bretzel, P. (1977). Concerning a probabilistic theory of causation adequate for the causal theory of time. *Synthese*, 35, 173 – 190.
- Bumcrot, R.J. (1964). Betweenness geometry in lattices. *Rendiconti del Circolo Matematico di Palermo*, 13, 11 – 28.
- Dowe P. (1992). Process causality and asymmetry. *Erkenntnis*, 37, 179 – 196.
- Ellett, Jr., F.S. & Ericson, D.P. (1986). Correlation, partial correlation, and causation. *Synthese*, 67, 157 – 173.



- Garey, M.R. & Johnson, D.S. (1979). *Computers and Intractability: A Guide to the Theory of NP-Completeness*. (San Francisco: W.H. Freeman and Company.)
- Hashimoto, J. (1958). Betweenness geometry. *Osaka Mathematical Journal*, 10, 147 – 158.
- Korb, K.B. (1999). Probabilistic causal structure. (In H. Sankey (Ed.), *Causation and Laws of Nature* (pp. 265 – 311). The Netherlands: Kluwer Academic Publishers.)
- Opatrný, J. (1979). Total ordering problem. *SIAM Journal on Computing*, 8, 111 – 114.
- Pitcher, E. & Smiley, M.F. (1942). Transitivity of betweenness. *Transactions of the American Mathematical Society*, 52, 95 – 114.
- Reichenbach, H. (1956). *The Direction of Time*. (Berkeley and Los Angeles: University of California Press.)
- Smiley, M.F. (1943). A comparison of algebraic, metric, and lattice betweenness. *Bulletin of the American Mathematical Society*, 49, 246 – 252.
- Weber, G.D. (1997). Discovering causal relations by experimentation: Causal trees. (In *Proceedings of the Eighth Midwest AI and Cognitive Science Conference* (pp. 91 – 98). Menlo Park: Association for the Advancement of Artificial Intelligence.)